数理解析研究所講究録
第 1798 巻 2012 年 121-133

Notes on the velocity-pressure relations in the incompressible fluids

Dongho Chae
Department of Mathematics
Chung-Ang University
Dongjak-gu Heukseok-ro 84
Seoul 156-756, Republic of Korea
e-mail: dchae@cau.ac.kr

Abstract

In this brief note we first show that a general integrable tensor satisfying the double divergence free equation has vanishing integrals on the diagonal components. This general theorem has applications to both of the compressible and the incompressible fluid equations. In particular it leads to pressure conditions leading to the vanishing of the velocity in the various fluid equations. In the second section we derive a formula representing average integrals of the pressure in terms of the integrals of the velocity components in the incompressible fluid equations.

AMS subject classification: 35Q35, 76B03
Key Words: Euler equations, Navier-Stokes equations, Liouville type results

1 Double divergence free tensors

Here we are concerned on the following *double divergence free equation* satisfied by $T = (T_{jk})$.

$$\sum_{j,k=1}^{N} \partial_j \partial_k T_{jk} = 0, \tag{1.1}$$

We present examples of

(i) The incompressible Euler equations in $\mathbb{R}^N$:

$$(E)\begin{cases} \dfrac{\partial v}{\partial t} + (v \cdot \nabla)v = -\nabla p \\ \operatorname{div}\ v = 0, \end{cases} \tag{1.2}$$

where $v = (v^1, \cdots v^N)$, $v^j = v^j(x,t)$, $j = 1, \cdots, N$, is the fluid velocity, and $p = p(x,t)$ is the pressure. Taking divergence operation on (E), we have the well-known velocity-pressure relation,

$$\sum_{j,k=1}^{N} \partial_j \partial_k (v_j v_k) = -\Delta p,$$

which is (1.1) with $T_{jk}(x,t) = v^j(x,t)v^k(x,t) + p(x,t)\delta_{jk}$

(ii) The compressible Euler equations:

$$(CE)\begin{cases} \partial_t \rho + \operatorname{div}(\rho v) = 0, \\ \partial_t(\rho v) + \operatorname{div}(\rho v \otimes v) = -\nabla p, \\ p = a\rho^\gamma, \end{cases}$$

where $\rho = \rho(x,t)$ is a density. In the stationary case the system (CE) can be written in the form of (1.1) with $T_{jk}(x) = \rho(x)v^j(x)v^k(x) + p(x)\delta_{jk}, p = a\rho^\gamma$.

(iii) In the classical field theories in the Minkowski space, $(R^{N+1}, \eta), \eta = \operatorname{diag}(-1, 1 \cdots 1)$, many of the *stationary* field equations can be written in the form,

$$\sum_{j=1}^{N} \partial_j T_{jk} = 0$$

for all $k = 1, \cdots, N$. Taking the divergence operation of this equation with respect k, we obtain (1.1).

Here is an implication of the $L^1(\mathbb{R}^N)$ condition on T_{jk}, solving (1.1).

Theorem 1.1 *Let $T \in L^1(\mathbb{R}^N)$. Then, we have*

$$\int_{\mathbb{R}^{N-1}} T_{jk}(x)\, d\mathbf{x}'_j = 0 \qquad \forall j, k = 1, \cdots, N \tag{1.3}$$

almost everywhere on $\mathbb{R}(dx_j)$, where we denoted

$$d\mathbf{x}'_j = dx_1 \cdots dx_{j-1} dx_{j+1} \cdots dx_N.$$

Proof In the weak formulation,

$$\sum_{j,k=1}^{N} \int_{\mathbb{R}^N} T_{jk} \partial_j \partial_k w(x)\, dx = 0 \quad \forall w \in C_0^\infty(\mathbb{R}^N),$$

We choose $w(x) = e^{i\xi_m x_m} \sigma_R(x)$, where $\sigma_R \in C_0^\infty(\mathbb{R}^N)$ is the previous cut-off. After taking $R \to \infty$, we obtain

$$\begin{aligned} 0 &= -\sum_{j,k=1}^{N} \int_{\mathbb{R}^N} T_{jk} \partial_j \partial_k (e^{i\xi_m x_m})\, dx = \xi_m^2 \int_{\mathbb{R}^N} T_{mm} e^{i\xi_m x_m} dx \\ &= \xi_m^2 \int_{-\infty}^{+\infty} \left\{ \int_{\mathbb{R}^{N-1}} T_{mm} d\mathbf{x}'_m \right\} e^{i\xi_m x_m} dx_m \\ &= \xi_m^2 \hat{f}(\xi_m), \qquad \text{where } f(x_m) := \textstyle\int_{\mathbb{R}^{N-1}} T_{mm}(x) d\mathbf{x}'_m. \end{aligned}$$

Hence, $\hat{f}(\xi_m) = 0 \quad \forall \xi_m \neq 0$. Since $\hat{f} \in C_0(\mathbb{R})(f \in L^1(\mathbb{R}^N))$, we extend by continuity that $\hat{f}(\xi_m) = 0$ for all $\xi_m \in \mathbb{R}$. Therefore, $f(x_m) = 0$ for all $x_m \in \mathbb{R}$. □

Corollary 1.1 *Let (v, p) solves (E), and satisfies $|v|^2 + |p| \in L^1(\mathbb{R}^N)$. Then, we have*

$$\int_{\mathbb{R}^{N-1}} v^j(x,t) v^k(x,t)\, d\mathbf{x}'_j = -\delta_{jk} \int_{\mathbb{R}^{N-1}} p(x,t)\, d\mathbf{x}'_j \tag{1.4}$$

for all $j, k = 1, \cdots, N$, and almost everywhere on $\mathbb{R}(dx_j)$.

Remark 1.1 Note that the above corollary is sharper than the result derived in [1](see also [3]).

Remark 1.2 Applying Theorem 1.1 to the compressible Euler equations we easily obtain that any stationary weak solution to the compressible Euler equations having finite energy corresponds to the vacuum, $\rho = 0$(see [4]).

One immediate consequence of the above corollary is the following.

Corollary 1.2 *For all $a \in \mathbb{R}^N$ and $b \in \mathbb{R}$ we have*

$$L^{N-1}\left(\{x \in \mathbb{R}^N \,|\, p(x) \leq 0\} \cap \{x \in \mathbb{R}^N \,|\, a \cdot x = b\}\right) > 0,$$

where $L^{N-1}(\cdot)$ denotes the Lebesgue measure on the hypersurface in $\mathbb{R}^N$ defined by $a \cdot x = b$. Namely the set S, where the $p(x)$ is non-positive, intersects with every hyperplane in $\mathbb{R}^N$.

The following result is can be regarded as a "spherical" version of Theorem 1.1.

Theorem 1.2 *Let $T \in L^1(\mathbb{R}^N, (1+|x|)^{-1}dx)$. Then, we have*

$$\int_{\mathbb{R}^N} \left\{ \left[\frac{tr(T)}{|x|} - \frac{x \cdot T \cdot x}{|x|^3} \right] w'(|x|) + \frac{x \cdot T \cdot x}{|x|^2} w''(|x|) \right\} dx = 0. \qquad (1.5)$$

for all radial function $w(x) = w(|x|)$ with

$$\frac{w'(r)}{r} + w''(r) \leq \frac{C}{1+r}. \qquad (1.6)$$

The proof follows from (2.2), choosing the radial test function $w(x) = w(|x|)$.

Corollary 1.3 *Let $T = (T_{jk})$ is a symmetric, positive definite tensor with $T \in L^1(\mathbb{R}^N, (1+|x|)^{-1}dx)$, which satisfies (1.1)). Then $T \equiv 0$ on $\mathbb{R}^N$.*

Next we specifically consider the incompressible Navier-Stokes and Euler equations.

Theorem 1.3 *Let (v, p) be a solution to the incompressible Navier-Stokes or Euler equations, which satisfies*

$$|v|^2 + |p| \in L^q(\mathbb{R}^N) \qquad \text{for some} \quad q \in (1, \frac{N}{N-1}).$$

Then, for all $r_0 \geq 0$, and $\eta > 0$ there holds the equality,

$$
\begin{aligned}
&\int_{r_0 \leq |x| \leq r_0+\eta} \left[N - \frac{r_0}{|x|}(N-1) \right] p(x) + (N-1)\eta \int_{|x| \geq r_0+\eta} p(x) dx \\
&= - \int_{r_0 \leq |x| \leq r_0+\eta} \left[\left(1 - \frac{r_0}{|x|}\right) |v|^2 + \frac{r_0}{|x|}(v \cdot x)^2 \right] dx \\
&\qquad -\eta \int_{|x| \geq r_0+\eta} \left[\frac{|v|^2}{|x|} - \frac{(v \cdot x)^2}{|x|} \right] dx.
\end{aligned} \tag{1.7}
$$

Proof We choose $w(x) = w(|x|)$ as follows.

$$
w(x) = \begin{cases} 0, & 0 \leq |x| \leq r_0 \\ \frac{1}{2}(|x| - r_0)^2, & r_0 \leq |x| \leq r_0 + \eta \\ \eta(|x| - r_0 - \eta) + \frac{\eta^2}{2}. & |x| \geq r_0 + \eta \end{cases}
$$

Then, we compute

$$
\partial_j \partial_k w(x) = \begin{cases} 0, & 0 \leq |x| \leq r_0 \\ \left(1 - \frac{r_0}{|x|}\right) \delta_{jk} + \frac{r_0 x_j x_k}{|x|^3}, & r_0 \leq |x| \leq r_0 + \eta \\ \eta \left(\frac{\delta_{jk}}{|x|} - \frac{x_j x_k}{|x|^3} \right). & |x| \geq r_0 + \eta \end{cases} \tag{1.8}
$$

Substituting w into (2.2), we obtain (1.7). □

Similar result to the above theorem is derived in [2] by a different argument. The following two corollaries are an immediate consequences of Theorem 1.3.

Corollary 1.4 *For all $R > 0$ we have $L^N\left(\{x \in \mathbb{R}^N \,|\, p(x) \leq 0,\ |x| > R\}\right) > 0$.*

Corollary 1.5 *Let $p(x,t)$ is the pressure corresponding to the nonzero velocity $v \in L^\infty(0,T; L^2(\mathbb{R}^N)) \cap L^2(0,T; H^1(\mathbb{R}^N))$ of a Leray-Hopf weak solution the Navier-Stokes equations. Then, for almost every $t \in (0,T)$ the set $\{x \in \mathbb{R}^N \,|\, p(x,t) \leq 0\}$ is unbounded.*

2 On the Navier-Stokes equations

In this section we concentrate on the Navier-Stokes equations(the Euler equations for $\nu = 0$) on $\mathbb{R}^3$.

$$\frac{\partial v}{\partial t} + (v \cdot \nabla)v = -\nabla p + \nu \Delta v \quad (x,t) \in \mathbb{R}^3 \times (0,T) \tag{2.1}$$

$$\text{div } v = 0, \quad (x,t) \in \mathbb{R}^3 \times (0,T) \tag{2.2}$$

Below we derive a formula representing a pressure average integral in terms of the velocity integrals using the spherical coordinates and cylindrical coordinates respectively. These are different approaches to a similar formula derived in [2]. We first use the representation of the velocity field in terms of the spherical coordinates, which is defined by

$$v = v_r e_r + v_\theta e_\theta + v_\phi e_\phi,$$

where

$$\begin{aligned} e_r &= (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \\ e_\theta &= (\cos\theta\cos\phi. \cos\theta\sin\phi, -\sin\theta) \\ e_\phi &= (-\sin\phi, \cos\phi, 0). \end{aligned}$$

In the theorem below we use the following notation of the smooth cut-off function.

$$\chi_{R,\delta}(r) = \begin{cases} 0 & \text{if } r \le R \\ 1 & \text{if } r \ge R+\delta, \end{cases}$$

and, *monotone increasing* on $(R, R+\delta)$.

Theorem 2.1 *If v is a smooth solution of the system (2.1)-(2.2) in $\mathbb{R}^3 \times (0,T)$, then the following equalities hold for all $t \in [0,T)$.*

(i) If $v \in C([0,T); L^q(\mathbb{R}^3))$ with $2 < q < 3$, then

$$\begin{aligned} &\int_{|x|>R} \left(\frac{|v|^2}{|x|} - \frac{v_r^2}{|x|} \right) \chi_{R,\delta}(|x|)dx + \int_{R<|x|<R+\delta} v_r^2 \partial_r \chi_{R,\delta}(|x|)dx \\ &\quad = -2\int_{|x|>R} \frac{p}{|x|}\chi_{R,\delta}(|x|)dx - \int_{R<|x|<R+\delta} p\, \partial_r \chi_{R,\delta}(|x|)dx \end{aligned} \tag{2.3}$$

for all $R, \delta > 0$.

(ii) If there exists a sequence $\{R_k\}_{k\in\mathbb{N}}$ with $R_k \uparrow \infty$ as $k \to \infty$ such that

$$\lim_{k\to\infty} \frac{1}{R_k^2} \int_{\partial B(0,R_k)} (p + v_r^2) dS = 0, \tag{2.4}$$

then

$$\int_{\mathbb{R}^3} \left(\frac{|v|^2}{|x|^3} - 3\frac{v_r^2}{|x|^3} \right) dy = -p(0,t) - \lim_{R\to 0} \frac{1}{R^2} \int_{\partial B(0,R)} v_r^2 dS. \tag{2.5}$$

Proof The system (2.1)-(2.2), in terms of the spherical coordinates is written as follows.

$$\begin{aligned}
\partial_t v_r + \left(v_r \partial_r + \frac{v_\theta}{r}\partial_\theta + \frac{v_\phi}{r\sin\theta}\partial_\phi \right) v_r &= \frac{v_\theta^2}{r} + \frac{v_\phi^2}{r} - \partial_r p + \nu(\Delta v)_r, \qquad (2.6)\\
\partial_t v_\theta + \left(v_r \partial_r + \frac{v_\theta}{r}\partial_\theta + \frac{v_\phi}{r\sin\theta}\partial_\phi \right) v_\theta &= -\frac{v_r v_\theta}{r} + \frac{v_\phi^2}{r}\cot\theta \\
&\quad - \frac{\partial_\theta p}{r} + \nu(\Delta v)_\theta, \qquad (2.7)\\
\partial_t v_\phi + \left(v_r \partial_r + \frac{v_\theta}{r}\partial_\theta + \frac{v_\phi}{r\sin\theta}\partial_\phi \right) v_\phi &= -\frac{v_r v_\phi}{r} - \frac{v_\theta v_\phi}{r}\cot\theta \\
&\quad - \frac{\partial_\phi p}{r\sin\theta} + \nu(\Delta v)_\phi, \qquad (2.8)
\end{aligned}$$

$$\frac{1}{r^2}\partial_r(r^2 v_r) + \frac{1}{r\sin\theta}\partial_\theta(\sin\theta v_\theta) + \frac{1}{r\sin\theta}\partial_\phi v_\phi = 0, \tag{2.9}$$

where

$$\begin{aligned}
(\Delta v)_r &= \Delta v_r - \frac{3v_r}{r^2} - \frac{2}{r^2}\partial_\theta v_\theta - \frac{2\cot\theta v_\theta}{r^2} - \frac{2}{r^2\sin\theta}\partial_\theta v_\theta,\\
(\Delta v)_\theta &= \Delta v_\theta + \frac{2}{r^2}\partial_\theta v_r - \frac{v_\theta}{r^2\sin^2\theta} - \frac{2\cos\theta}{r^2\sin^2\theta}\partial_\phi v_\phi,\\
(\Delta v)_\phi &= \Delta v_\phi - \frac{v_\phi}{r^2\sin^2\theta} + \frac{2}{r^2\sin\theta}\partial_\phi v_r + \frac{2\cos\theta}{r^2\sin^2\theta}\partial_\phi v_\theta. \qquad (2.10)
\end{aligned}$$

Let $B(0,r) = \{x \in \mathbb{R}^3 \,|\, |x| < r\}$. We integrate (2.6) over $\partial B(0,r)$, then

$$\begin{aligned}
&\partial_t \int_{\partial B(0,r)} v_r\, dS + \int_{\partial B(0,r)} \left(v_r\partial_r + \frac{v_\theta}{r}\partial_\theta + \frac{v_\phi}{r\sin\theta}\partial_\phi \right) v_r\, dS \\
&= \int_{\partial B(0,r)} \left(\frac{v_\theta^2}{r} + \frac{v_\phi^2}{r} - \partial_r p \right) dS + \nu \int_{\partial B(0,r)} (\Delta v)_r dS. \qquad (2.11)
\end{aligned}$$

For the first term of the left hand side of (2.11) we have

$$\partial_t \int_{\partial B(0,r)} v_r \, dS = \partial_t \int_{B(0,r)} \operatorname{div} v \, dy = 0 \tag{2.12}$$

by the divergence theorem. Using (2.9), we can write the second term of the left hand side of (2.11)

$$\begin{aligned}
&\int_{\partial B(0,r)} \left(v_r \partial_r + \frac{v_\theta}{r}\partial_\theta + \frac{v_\phi}{r\sin\theta}\partial_\phi \right) v_r \, dS \\
&= \int_{\partial B(0,r)} \left\{ \frac{1}{r^2}\partial_r(r^2 v_r^2) + \frac{1}{r\sin\theta}\partial_\theta(v_\theta v_r \sin\theta) + \frac{1}{r\sin\theta}\partial_\phi(v_\phi v_r) \right\} dS \\
&= \int_{\mathbb{S}^2} \left\{ \frac{1}{r^2}\partial_r(r^2 v_r^2) + \frac{1}{r\sin\theta}\partial_\theta(v_\theta v_r \sin\theta) + \frac{1}{r\sin\theta}\partial_\phi(v_\phi v_r) \right\} r^2 d\Sigma \\
&= \int_{\mathbb{S}^2} \partial_r(r^2 v_r^2) d\Sigma,
\end{aligned} \tag{2.13}$$

where we set $d\Sigma = \sin\theta d\theta d\phi$. The viscosity term of (2.11) vanishes, since

$$\int_{\partial B(0,r)} (\Delta v)_r dS = \int_{B(0,r)} \operatorname{div}(\Delta v) dy = 0 \tag{2.14}$$

by the divergence theorem, and the divergence free condition for v. Taking into account (2.12)-(2.14), we obtain from (2.11) that

$$\int_{\partial B(0,r)} \left(\frac{v_\theta^2}{r} + \frac{v_\phi^2}{r} \right) dS = \int_{\mathbb{S}^2} \partial_r(r^2 v_r^2) d\Sigma + \int_{\partial B(0,r)} \partial_r p dS \tag{2.15}$$

Let us introduce a radial cut-off function $\sigma \in C_0^\infty(\mathbb{R}^3)$ such that

$$\sigma(r) = \begin{cases} 1 & \text{if } r < 1 \\ 0 & \text{if } r > 2, \end{cases} \tag{2.16}$$

and $0 \le \sigma(r) \le 1$ for $1 < r < 2$. Then, for each $R, R_1 > 0$, we define

$$\sigma_R(r) = \sigma\left(\frac{r}{R}\right) \tag{2.17}$$

Below we fix $R > 0$ and choose $R_1 > 2R$. Multiplying (2.15) by $\sigma_{R_1}(r)\chi_{R,\delta}(r)$ and integrating it with respect r over $(0,\infty)$, we have

$$
\begin{aligned}
&\int_{\mathbb{R}^3}\left(\frac{v_\theta^2}{r}+\frac{v_\phi^2}{r}\right)\sigma_{R_1}(r)\chi_{R,\delta}(r)dx = \int_0^\infty\int_{\mathbb{S}^2}\partial_r(r^2v_r^2\sigma_{R_1}(r)\chi_{R,\delta}(r))d\Sigma dr\\
&\quad -\int_0^\infty\int_{\mathbb{S}^2}r^2v_r^2\chi_{R,\delta}(r)\partial_r\sigma_{R_1}(r)d\Sigma dr-\int_0^\infty\int_{\mathbb{S}^2}r^2v_r^2\sigma_{R_1}(r)\partial_r\chi_{R,\delta}(r)d\Sigma dr\\
&\quad +\int_0^\infty\int_{\mathbb{S}^2}\sigma_{R_1}(r)\chi_{R,\delta}(r)\partial_r p\, r^2 d\Sigma dr\\
&\quad = -\int_{\mathbb{R}^3}v_r^2\chi_{R,\delta}(r)\partial_r\sigma_{R_1}(r)dx-\int_{\mathbb{R}^3}v_r^2\sigma_{R_1}(r)\partial_r\chi_{R,\delta}(r)dx\\
&\quad -2\int_{\mathbb{R}^3}\frac{\sigma_{R_1}(r)\chi_{R,\delta}(r)}{r}p\,dx-\int_{\mathbb{R}^3}p\chi_{R,\delta}(r)\partial_r\sigma_{R_1}(r)\,dx\\
&\quad -\int_{\mathbb{R}^3}p\sigma_{R_1}(r)\partial_r\chi_{R,\delta}(r)\,dx\\
&\quad := I_1+\cdots+I_5 \qquad\qquad (2.18)
\end{aligned}
$$

after integration by part. We estimate

$$
\begin{aligned}
|I_1| &\le \frac{1}{R_1}\int_{R_1\le|x|\le 2R_1}v_r^2|\sigma'(\frac{r}{R_1})|dx\\
&\le \frac{\|\sigma'\|_{L^\infty}}{R_1}\left(\int_{R_1\le|x|\le 2R_1}|v_r|^q dx\right)^{\frac{2}{q}}\left(\int_{R_1\le|x|\le 2R_1}1dx\right)^{\frac{q-2}{q}}\\
&\le CR_1^{\frac{2(q-3)}{q}}\|v\|_{L^q}^2\to 0 \qquad\qquad (2.19)
\end{aligned}
$$

as $R_1\to\infty$ since $v\in L^q(\mathbb{R}^3)$ for $2<q<3$.

$$
\begin{aligned}
|I_4| &\le \frac{1}{R_1}\int_{R_1\le|x|\le 2R_1}|p||\sigma'(\frac{r}{R})|dx \le \frac{C}{R_1}\left(\int_{\mathbb{R}^3}|p|^{\frac{q}{2}}dx\right)^{\frac{2}{q}}\|\sigma'\|_{L^\infty}R_1^{\frac{3(q-2)}{q}}\\
&\le CR_1^{\frac{2(q-3)}{q}}\|v\|_{L^q}^2\to 0 \qquad\qquad (2.20)
\end{aligned}
$$

as $R_1\to\infty$ for $2<q<3$. On the other hand, by the dominated convergence theorem we easily find

$$
I_2\to-\int_{\mathbb{R}^3}v_r^2\partial_r\chi_{R,\delta}(r)dx,\quad I_3\to -2\int_{\mathbb{R}^3}\frac{\chi_{R,\delta}(r)}{r}p\,dx,\quad I_5\to-\int_{\mathbb{R}^3}p\partial_r\chi_{R,\delta}(r)\,dx \qquad (2.21)
$$

as $R_1 \to \infty$. Hence, passing $R_1 \to \infty$ in (2.15), and using the fact $v_\theta^2 + v_\phi^2 = |v|^2 - v_r^2$, we get (2.3). In order to prove (2.24) we rewrite (2.15) in the form

$$\int_{\partial B(0,r)} \left(\frac{v_\theta^2}{r^3} + \frac{v_\phi^2}{r^3} - 2\frac{v_r^2}{r^3} \right) dS = \partial_r \int_{\mathbb{S}^2} (p + v_r^2) d\Sigma. \tag{2.22}$$

Integrating (2.22) over $(0, R_k)$ with respect to r, and passing $k \to \infty$, we obtain (2.24). □

Given $R, h \in (0, \infty]$ we define a cylinder $\mathcal{C}_{R,h} := \{x \in \mathbb{R}^3 \,|\, x_1^2 + x_2^2 < R,\ -h < x_3 < h\}$, and we denote its boundaries as

$$\partial \mathcal{C}_{R,h} = \mathcal{B}_{R,h} \cup \mathcal{S}_{R,h}, \tag{2.23}$$

where $\mathcal{B}_{R,h} = \{x \in \mathbb{R}^3 \,|\, x_1^2 + x_2^2 < R,\ x_3 = \pm h\}$ is the upper and lower bases, and $\mathcal{S}_{R,h} = \{x \in \mathbb{R}^3 \,|\, x_1^2 + x_2^2 = R,\ -h < x_3 < h\}$ is the side. In the theorems below we use the representations in terms of the cylindrical coordinate,

$$v = v_r e_r + v_\phi e_\phi + v_3 e_3,$$

where

$$e_r = (\cos\phi, \sin\phi, 0),\ e_\phi = (-\sin\phi, \cos\phi, 0,),\ e_3 = (0, 0, 1).$$

Theorem 2.2 *Let (v, p) be a smooth solution of the system (2.1)-(2.2) on $\mathbb{R}^3 \times (0, T)$. We assume that*

$$|p| + |v|^2 \in L^1(\mathbb{R}^3).$$

Then,

$$\int_{-\infty}^{\infty} p(0, 0, x_3, t) dx_3 = \int_{\mathbb{R}^3} \left(\frac{v_r^2}{r^2} - \frac{v_\phi^2}{r^2} \right) dx. \tag{2.24}$$

Proof The system (2.1)-(2.2) can be written as

$$\partial_t v_r + (v_r \partial_r + \frac{v_\phi}{r} \partial_\phi + v_3 \partial_3) v_r = \frac{v_\phi^2}{r} - \partial_r p + \nu (\Delta v)_r, \tag{2.25}$$

$$\partial_t v_\phi + (v_r \partial_r + \frac{v_\phi}{r} \partial_\phi + v_3 \partial_3) v_\phi = -\frac{v_\phi v_r}{r} - \frac{1}{r} \partial_r p + \nu (\Delta v)_\phi, \tag{2.26}$$

$$\partial_t v_3 + (v_r \partial_r + \frac{v_\phi}{r} \partial_\phi + v_3 \partial_3) v_3 = -\partial_3 p + \nu (\Delta v)_3, \tag{2.27}$$

$$\frac{1}{r} \partial_r (r v_r) + \frac{1}{r} \partial_\phi v_\phi + \partial_3 v_3 = 0, \tag{2.28}$$

where

$$
\begin{aligned}
(\Delta v)_r &= \Delta v_r - \frac{2}{r^2}\partial_\phi v_\phi - \frac{v_r}{r^2}, \\
(\Delta v)_\phi &= \Delta v_\phi + \frac{2}{r^2}\partial_\phi v_r - \frac{v_\phi}{r^2}, \\
(\Delta v)_3 &= \Delta v_3. \qquad (2.29)
\end{aligned}
$$

Multiplying (2.25) by r, and integrating it with respect to (ϕ, z) over $(0, 2\pi)\times(-\infty, \infty)$, we have

$$
\begin{aligned}
&\int_{-\infty}^{\infty}\int_0^{2\pi} v_\phi^2 d\phi dx_3 - \int_{-\infty}^{\infty}\int_0^{2\pi} r\partial_r p\, d\phi dx_3 \\
&= \partial_t \int_{-\infty}^{\infty}\int_0^{2\pi} v_3 r\, d\phi dx_3 + \int_{-\infty}^{\infty}\int_0^{2\pi} \left\{(v_r\partial_r + \frac{v_\phi}{r}\partial_\phi + v_3\partial_3)v_r\right\} r\, d\phi dx_3 \\
&\quad -\nu \int_{-\infty}^{\infty}\int_0^{2\pi} (\Delta v)_r r\, d\phi dx_3 \\
&:= I_1 + I_2 + I_3. \qquad (2.30)
\end{aligned}
$$

We have by the divergence theorem,

$$
\begin{aligned}
\int_{-\infty}^{\infty}\int_0^{2\pi} v_r(r,\phi,x_3,t) r\, d\phi dx_3 &= \int_{-\infty}^{\infty}\int_0^{2\pi} v_r(r,\phi,x_3,t) r\, d\phi dx_3 \\
&\quad + \lim_{h\to\infty}\int_0^{2\pi}\int_0^r \{v_3(\rho,\phi,h) - v_3(\rho,\phi,-h)\}\,\rho d\rho d\phi \\
&= \lim_{h\to\infty}\int_{B(0,r)\times[-h,h]} \operatorname{div} v\, dy = 0. \qquad (2.31)
\end{aligned}
$$

Hence, $I_1 = 0$. Using the formula (2.28), we write

$$
\begin{aligned}
I_2 &= \int_{-\infty}^{\infty}\int_0^{2\pi} \left\{\frac{1}{r}\partial_r(rv_r^2) + \frac{1}{r}\partial_\phi(v_\phi v_r) + \partial_3(v_3 v_r)\right\} r\, d\phi dx_3 \\
&= \int_{-\infty}^{\infty}\int_0^{2\pi} \partial_r(rv_r^2) d\phi dx_3. \qquad (2.32)
\end{aligned}
$$

Similarly to (2.31) we have

$$\int_{-\infty}^{\infty}\int_0^{2\pi}(\Delta v)_r(r,\phi,x_3,t)r\,d\phi dx_3 = \int_{-\infty}^{\infty}\int_0^{2\pi}(\Delta v)_r(r,\phi,x_3,t)r\,d\phi dx_3$$
$$+\lim_{h\to\infty}\int_0^{2\pi}\int_0^{r}\left\{(\Delta v)_3(\rho,\phi,h)-(\Delta v)_3(\rho,\phi,-h)\right\}\rho d\rho d\phi$$
$$=\lim_{h\to\infty}\int_{B(0,r)\times[-h,h]}\mathrm{div}\,(\Delta v)\,dy=0, \tag{2.33}$$

and $I_3=0$. Hence one obtain,

$$\int_{-\infty}^{\infty}\int_0^{2\pi}v_\phi^2 d\phi dx_3 = \int_{-\infty}^{\infty}\int_0^{2\pi}\partial_r(rv_r^2)d\phi dx_3 + \int_{-\infty}^{\infty}\int_0^{2\pi}r\partial_r p\,d\phi dx_3$$
$$= r\int_{-\infty}^{\infty}\int_0^{2\pi}\partial_r(v_r^2)d\phi dx_3 + \int_{-\infty}^{\infty}\int_0^{2\pi}v_r^2 d\phi dx_3 + r\int_{-\infty}^{\infty}\int_0^{2\pi}\partial_r p\,d\phi dx_3,$$

and

$$\int_{-\infty}^{\infty}\int_0^{2\pi}\left(\frac{v_r^2}{r}-\frac{v_\phi^2}{r}\right)d\phi dx_3 = -\partial_r\int_{-\infty}^{\infty}\int_0^{2\pi}(p+v_r^2)d\phi dx_3. \tag{2.34}$$

Integrating (2.34) with respect to r over $(0,R_k)$, and passing $k\to\infty$, observing

$$\lim_{R\to 0}\frac{1}{R}\int_{S_{R,\infty}}v_r^2 dS=0,$$

due to the smoothness at $r=0$, and there exists a sequence $\{r_k\}\uparrow\infty$ such that

$$\int_{-\infty}^{\infty}\int_0^{2\pi}(p+v_r^2)d\phi dx_3\bigg|_{r=r_k}\to 0 \quad \text{as } k\to\infty$$

due to the hypothesis $|p|+|v|^2\in L^1(\mathbb{R}^3)$, we obtain (2.24). $\square$

If we consider the system (2.1)-(2.2) on the domain

$$\mathfrak{D}=\mathbb{R}^2\times\mathbb{T}=\{x\in\mathbb{R}^3\,|\,(x_1,x_2)\in\mathbb{R}^2,\ z\in(-L,L)\} \tag{2.35}$$

with the periodic boundary condition in the x_3- direction. The similar proof to the Theorem 2.2 leads to the following result.

Theorem 2.3 *Let* (v, p) *be a smooth solution of the system (2.1)-(2.2) on* $\mathfrak{D} \times (0, T)$. *We assume* $|p| + |v|^2 \in L^1(\mathfrak{D})$. *Then*

$$\int_{-\infty}^{\infty} p(0, 0, x_3, t) dx_3 = \int_{\mathfrak{D}} \left(\frac{v_r^2}{r^2} - \frac{v_\phi^2}{r^2} \right) dx. \tag{2.36}$$

Acknowledgements

This research was partially supported by NRF grant no.2006-0093854.

References

[1] L. Brandolese and Y. Meyer, *On the instantaneous spreading for the Navier-Stokes system in the whole space,* Contr. Optim. Calc. Var. **8**, (2002), pp. 273-285.

[2] D. Chae, *Conditions on the pressure for vanishing velocity in the incompressible fluid flows,* Comm. PDE, in press.

[3] D. Chae, *Liouville type of theorems for the Euler and the Navier-Stokes equations,* Advances in Math., **228**, (2011), pp. 2855-2868 .

[4] D. Chae, *On the nonexistence of global weak solutions to the Navier-Stokes-Poisson equations in* $\mathbb{R}^N$, Comm. PDE, **35**, no. 3, (2010), pp. 535-557.